#5 - tracking precision and correlations

If your ever need to design, calibrate, or use data from a tracking system, you must read and understand the paper by Gluckstern:

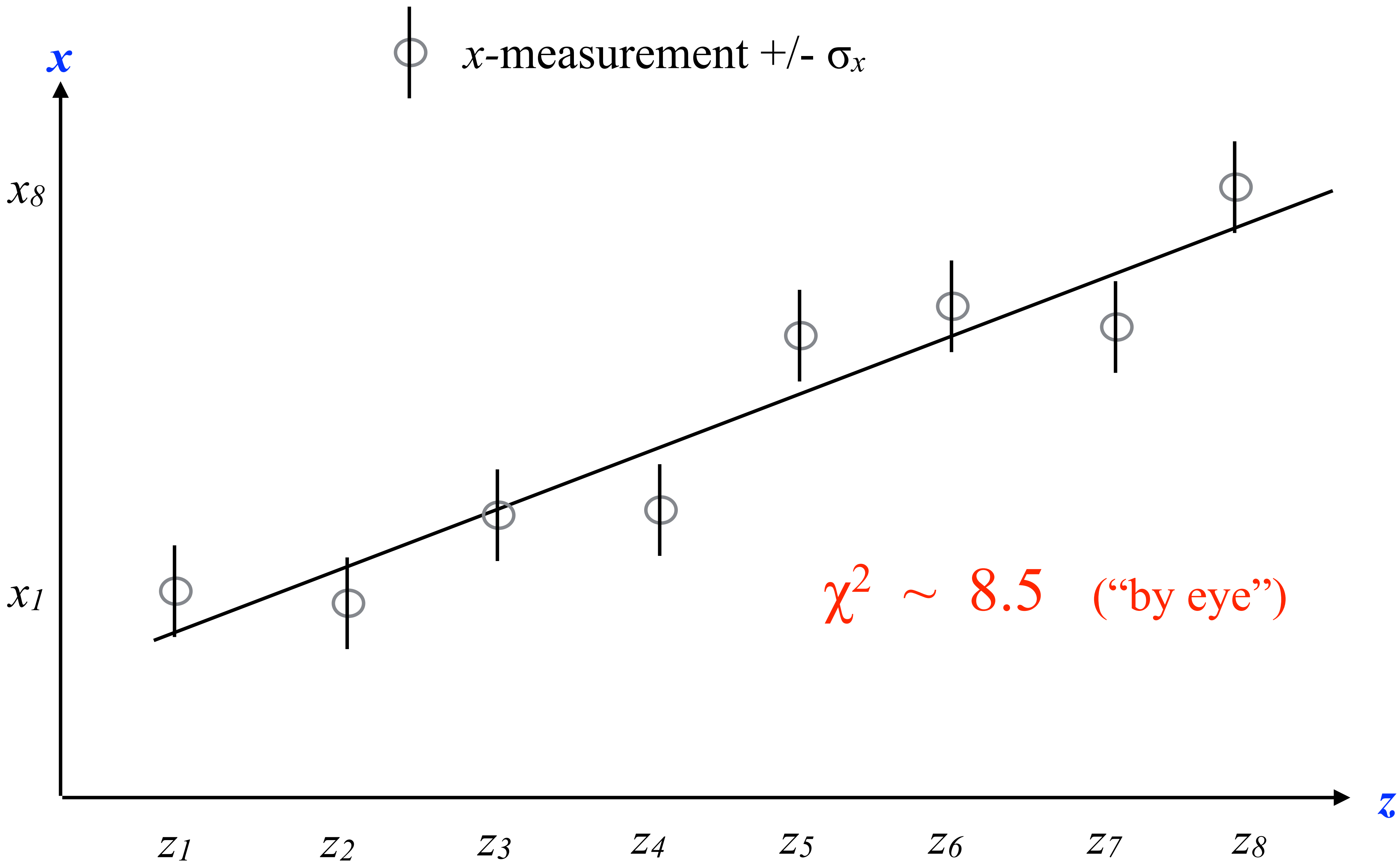
”Uncertainties in track momentum and direction, due to multiple scattering and measurement errors,” R. L. Gluckstern,

Nucl. Instr. Meth. **24** (1963) 381.

He starts from a simple parabolic track segment with transverse spatial measurements of the trajectory. I have this article on paper in my office, but not here ….

"Least squares" fit

$$\chi^2 = \sum_i \left[\frac{x_i(\text{meas}) - x_i(\text{theory})}{\sigma_i(\text{theory})}\right]^2$$



Least squares fit (to a straight line, $x = mz + b$):

N points on line; fit parameters are slope m and x-intercept b.

"chi-squared", χ^2, is the summed deviations of the measurements from the (theoretical) expectation, in units of the expected rms uncertainty per point, σ_i:

$$\chi^2 = \sum_i^N [\frac{x_i - (mz_i + b)}{\sigma_i}]^2$$

The "best estimates" of m and b are at the minimum in χ^2. So ...

$$0 = \frac{\partial \chi^2}{\partial m} \qquad \text{and} \qquad 0 = \frac{\partial \chi^2}{\partial b}$$

It is good to take out the factor of 2 so that the derivative is like the argument to a Gaussian:

$$0 = \frac{1}{2}\frac{\partial \chi^2}{\partial m} \qquad \text{and} \qquad 0 = \frac{1}{2}\frac{\partial \chi^2}{\partial b}$$

These derivatives give:

$$\sum \frac{x_i z_i}{\sigma_i^2} = m \sum \frac{z_i^2}{\sigma_i^2} + b \sum \frac{z_i}{\sigma_i^2}$$

and

$$\sum \frac{x_i}{\sigma_i^2} = m \sum \frac{z_i}{\sigma_i^2} + b \sum \frac{1}{\sigma_i^2}$$

Looks like linear equations -

$$\left(\begin{array}{c} s_m \\ s_b \end{array} \right) = \begin{bmatrix} M_{mm} & M_{mb} \\ M_{bm} & M_{bb} \end{bmatrix} \times \left(\begin{array}{c} m \\ b \end{array} \right)$$

Solution for m, b is

$$\left(\begin{array}{c} m \\ b \end{array} \right) = \begin{bmatrix} M_{mm} & M_{mb} \\ M_{bm} & M_{bb} \end{bmatrix}^{-1} \times \left(\begin{array}{c} s_m \\ s_b \end{array} \right)$$

The nice part is that the second derivatives give the terms in the error matrix:

$$\frac{1}{2}\frac{\partial^2\chi^2}{\partial m^2} = 1/\sigma_m^2$$

and

$$\frac{1}{2}\frac{\partial^2\chi^2}{\partial b^2} = 1/\sigma_b^2$$

and the cross derivatives $\frac{\partial^2\chi^2}{\partial m \partial b}$, etc., give the fitted correlations,

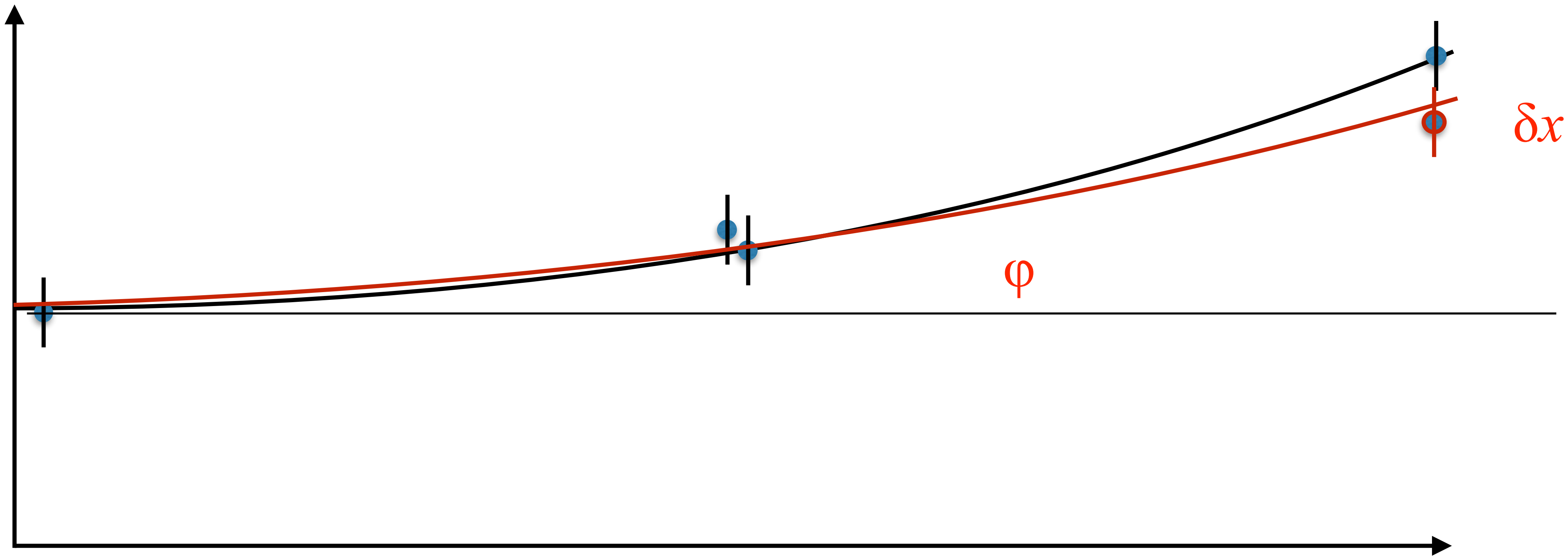
$$\frac{1}{2}\frac{\partial^2\chi^2}{\partial m \partial b} = \frac{1}{2}\frac{\partial^2\chi^2}{\partial b \partial m} = 1/\sigma_{mb}^2$$

So you have the complete error matrix for the fit.

Gluckstern did the same thing except he used

$$(b + mz + cz^2)$$

where his curvature term c was the inverse radius of curvature $c = 1/R \approx 1/p$.



An important correlation results from his fit: a fluctuation in the spatial measurement δx leads to a negative correlation between the momentum and the azimuthal angle of a track:

$$\overline{\delta p \delta \phi} \rightarrow \sigma^2_{p\phi} < 0$$

In this fit, $-\delta x$ leads to $+\delta p$ and $-\delta\phi$.

Back to tracking:

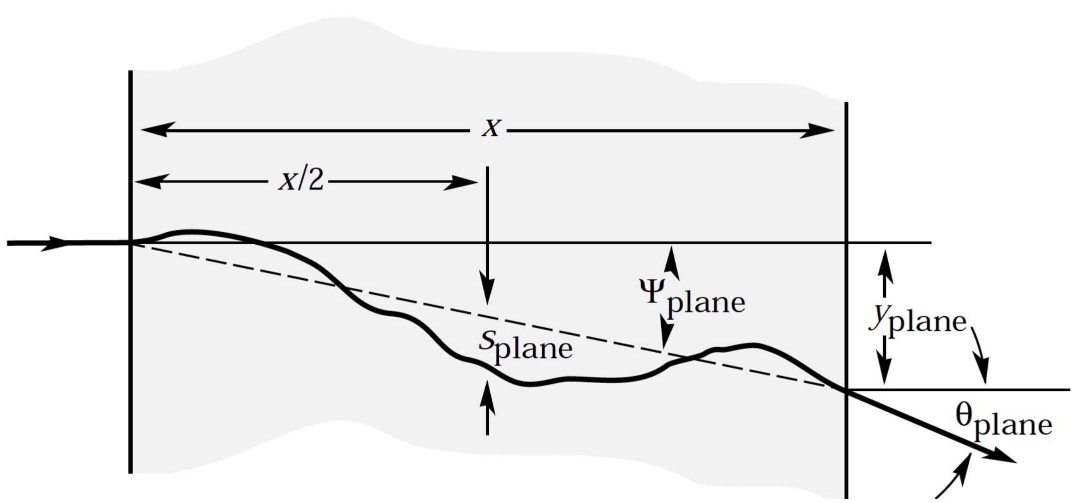
it is useful to express the momentum uncertainty in terms of the curvature, *k*, defined as the inverse radius of curvature, *k=1/R*.

The uncertainty in the curvature has two terms: one from the sagitta uncertainty and the other from the multiple scattering uncertainty.

curvature due to sagitta: $k_{\text{sagitta}} = \frac{8s}{\ell^2}$

curvature uncertainty: $\delta k_{\text{sagitta}} = \frac{8}{\ell^2}\delta x \rightarrow \frac{1}{\ell^2}\sqrt{\frac{720}{N+4}}\delta x$

curvature uncertainty due
to multiple scattering: $\delta k_{\text{MS}} = \frac{8}{\ell^2}s \rightarrow \frac{8}{\ell^2}s_{\text{rms}}$



ms $= \frac{\ell}{4\sqrt{3}}\theta_{\text{rms}}$

$$\delta k_{\text{MS}} = \frac{8}{\ell}\frac{\theta_{\text{rms}}}{4\sqrt{3}}$$

The full multiple scattering sagitta uncertainty is

$$\delta k_{\mathrm{MS}} = \frac{0.0157\mathrm{GeV/c}}{\ell p}\sqrt{\ell/X_0}[1 + 0.038\ln(\ell/X_0)]$$

$$\approx \theta_{\mathrm{rms}}/\ell$$

Overall momentum uncertainty (MS and sagitta independent):

$$\frac{\sigma_p}{p^2} = \frac{\sigma_x}{0.3B\ell^2}\sqrt{\frac{720}{N+4}} \oplus$$

$$\frac{0.0157\text{GeV/c}}{0.3B\ell p}\sqrt{\ell/X_0}[1 + 0.038\ln(\ell/X_0)]$$

This can be seen as:

$$\frac{\sigma_p}{p^2} = [\text{detector construction}] \oplus [\text{material budget}]/p$$

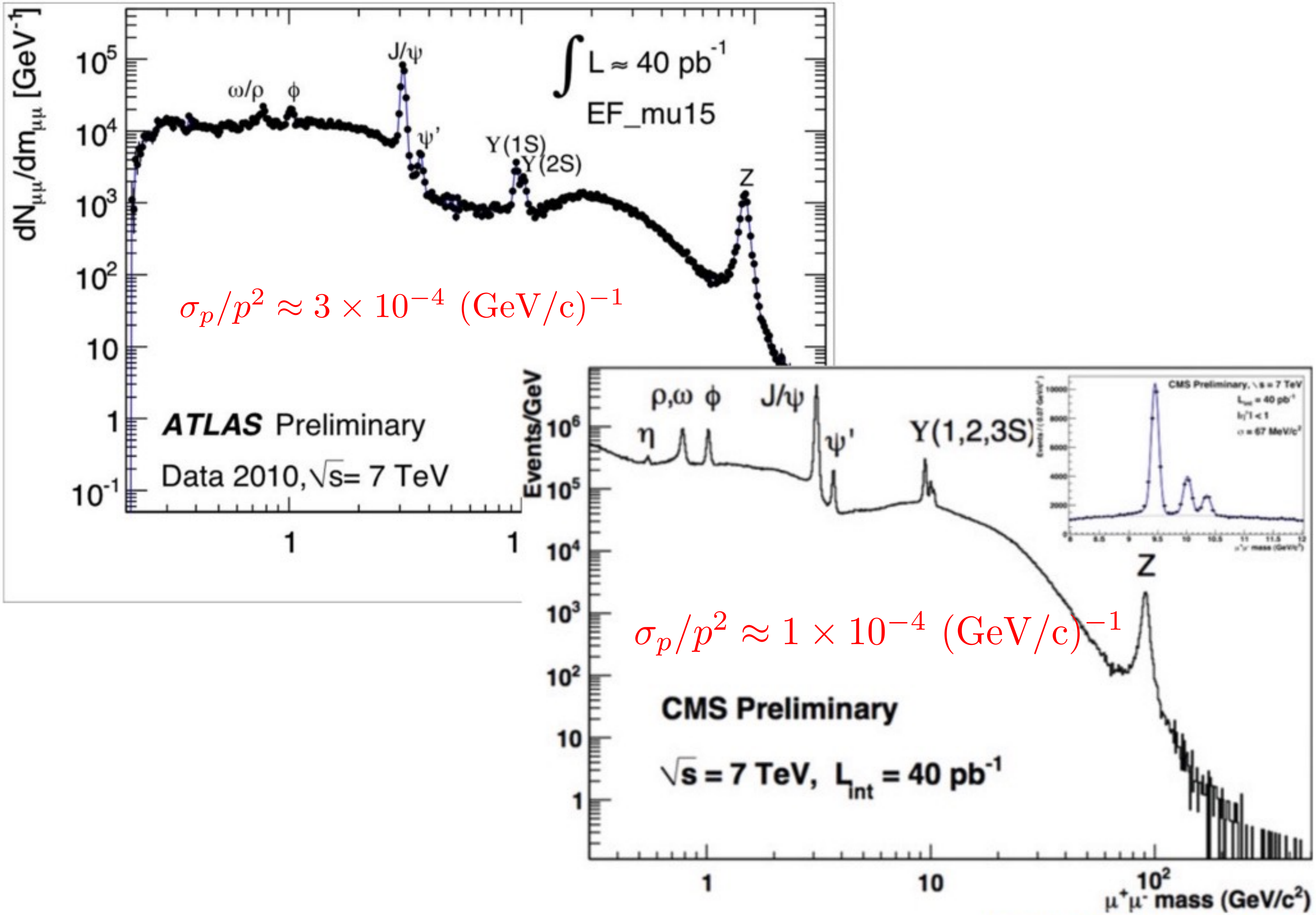
As a track points more in the "forward" direction in a solenoidal magnetic field,
$\theta \sim 0$,
two things happen:

i) the full radial track length is not reached as the particle enters the end cap, and
ii) the Lorentz bending force is less as v x B goes to zero.

Both effects can be represented and scaled as:

$$\sigma_p/p^2 \rightarrow 1/\sin^{5/2}\theta$$

You can see tracking resolution in ATLAS and CMS data:



$\sigma_p/p^2 \approx 3\times10^{-4}\ (\mathrm{GeV/c})^{-1}$ (ATLAS)

$\sigma_p/p^2 \approx 1\times10^{-4}\ (\mathrm{GeV/c})^{-1}$ (CMS)

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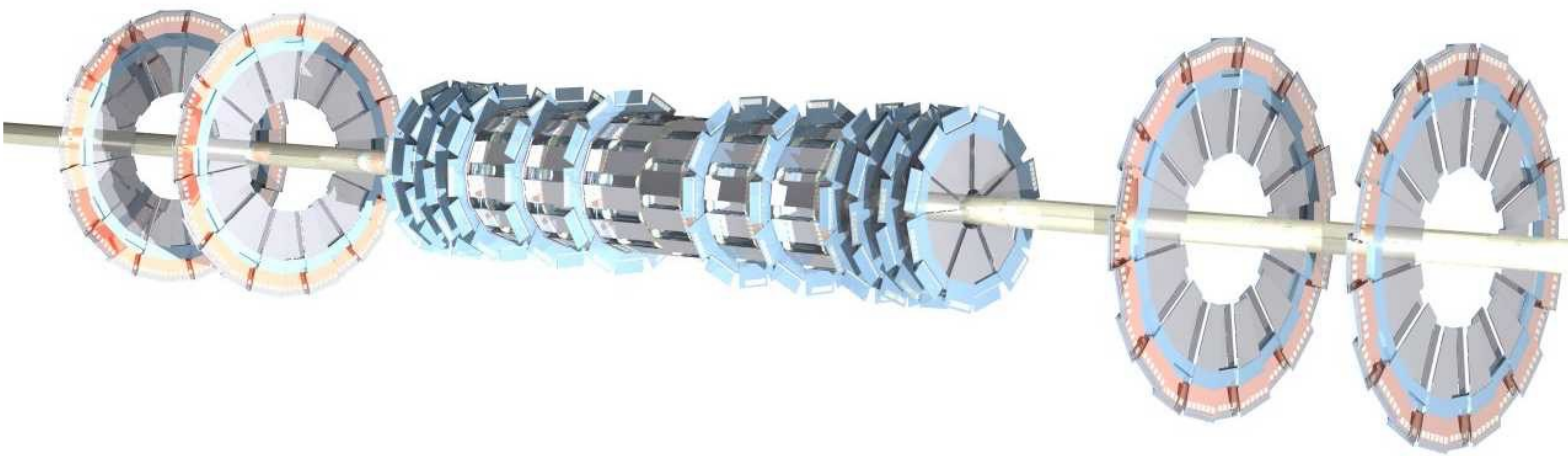
Vertex chamber for impact parameter measurement: e.g., D0



Impact parameter measurement: pure geometry, but not simple

Leverage:

$$\sigma_b^x \approx \xi \cdot \sigma_x \qquad \text{where} \quad \xi \approx \frac{r_1}{r_N - r_1} \frac{1}{\sqrt{N}}$$

Multiple scattering is driven simply by $\theta_{\rm rms}$

$$\sigma_b^{\rm MS} \approx \frac{r_1}{\sin\theta} \frac{0.0136}{p\beta} \sqrt{\frac{\ell/\sin\theta}{X_0}} [1 + 0.038 \ln(\ell/\sin\theta)/X_0]$$

Gathering the $\sin\theta$ and ln terms into ζ:

$$\zeta = \frac{r_1}{\sin^{3/2}\theta} [1 + 0.038 \ln(1/\sin\theta)],$$

the multiple scattering contribution to σ_b can be written as:

$$\sigma_b^{\rm MS} \approx \zeta \cdot \theta_{\rm rms}.$$

The third contribution is from the lateral displacement of the track due to multiple scattering, $y_{\rm rms} = r \cdot \theta_{\rm rms}/\sqrt{3}$, which scales like $1/p$ and comes from all the layers of the vertex chamber. Summing these in quadrature leads to a term like

$$\sigma_b \approx \eta/\sqrt{p}.$$

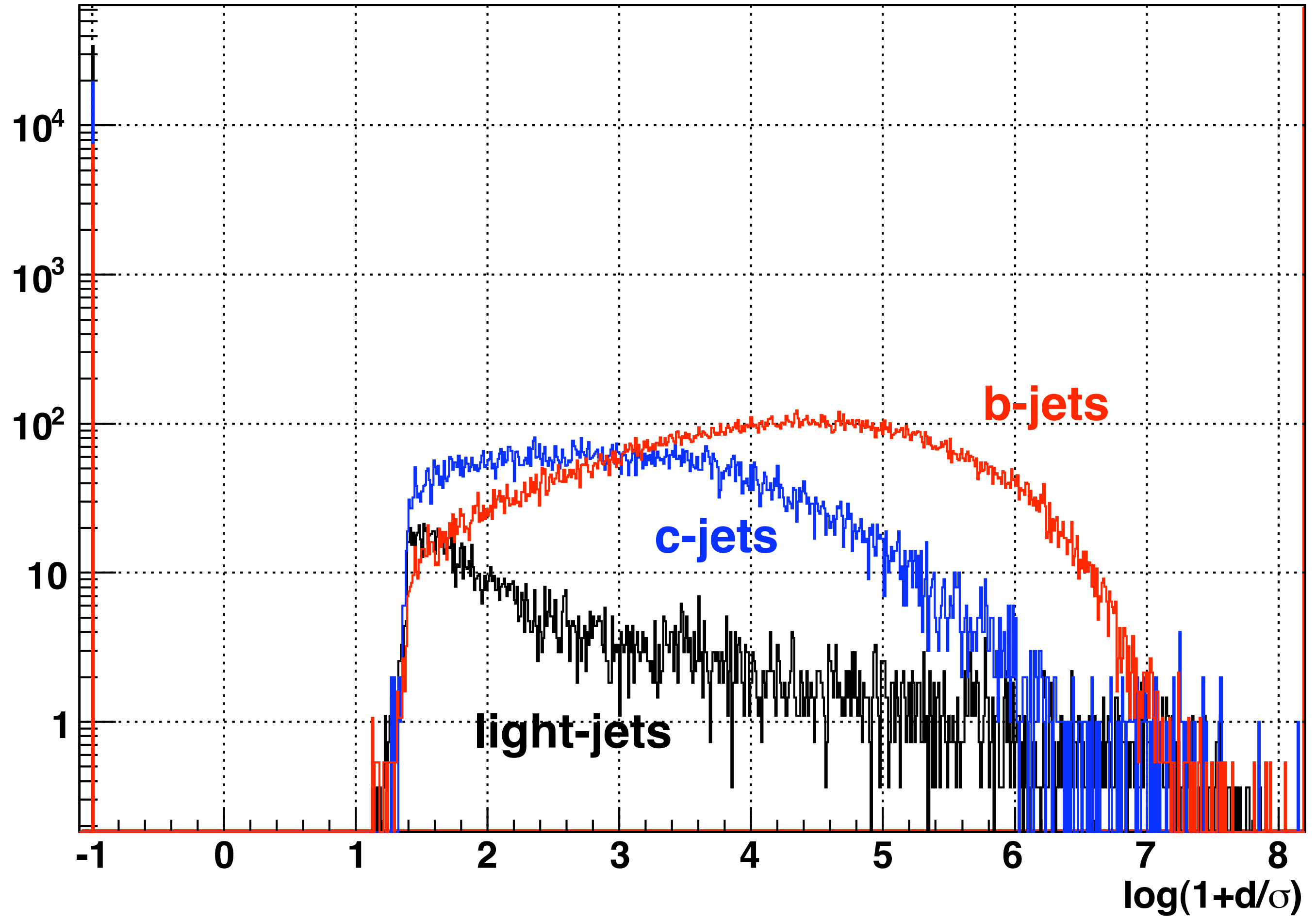
However, there are correlations between these terms [see PDG, Sec.27.3]. Simulation. The overall impact parameter resolution is

$$\sigma_b \approx (\xi \cdot \sigma_x) \oplus (\zeta \cdot \theta_{\rm rms}) \oplus (\eta/\sqrt{p})$$

Therefore, a better impact parameter resolution is achieved by

i) small r_1 (get close to the beam)
ii) small l/X_0 (low material budget)

Higher momentum always helps.



A goal for an e^+e^- collider is

$$\sigma_b \approx 5\mu\text{m} \oplus 10\mu\text{m}/(p\sin^{3/2}\theta) \oplus 10\mu\text{m}/\sqrt{p}$$

This can be achieved with $(20\mu\text{m})^2$ pixels, providing the mass is low enough.

We see that the momentum resolution goes like $\sim 1/l^2$. If a track goes from the integration vertex, through the vertex chamber, and through the tracking chamber, its length is larger.

However, this is nullified if there is a lot of multiple scattering between the vertex chamber and the main tracking chamber.

Summary of tracking:

- Inverse momentum, $1/p$, is Gaussian,
- Optimum momentum resolution has 1/4 of the measurements at either of a track segment, and 1/2 at the middle,
- The momentum and azimuth angle are negatively correlated,
- Helical tracks not perpendicular to the B-field, “dip” angle λ,

$$\delta k_{\mathrm{sagitta}} \to \delta k_{\mathrm{sagitta}} / \cos^2 \lambda \quad \text{and} \quad \delta k_{\mathrm{MS}} \to \delta k_{\mathrm{MS}} / \cos^2 \lambda$$

- Modern methods for tracking, e.g., Kalman Filter, do not alter the results from Gluckstern, and
- The impact parameter of tracks from a 2-body decay are approximately independent of the parent particle momentum.

History

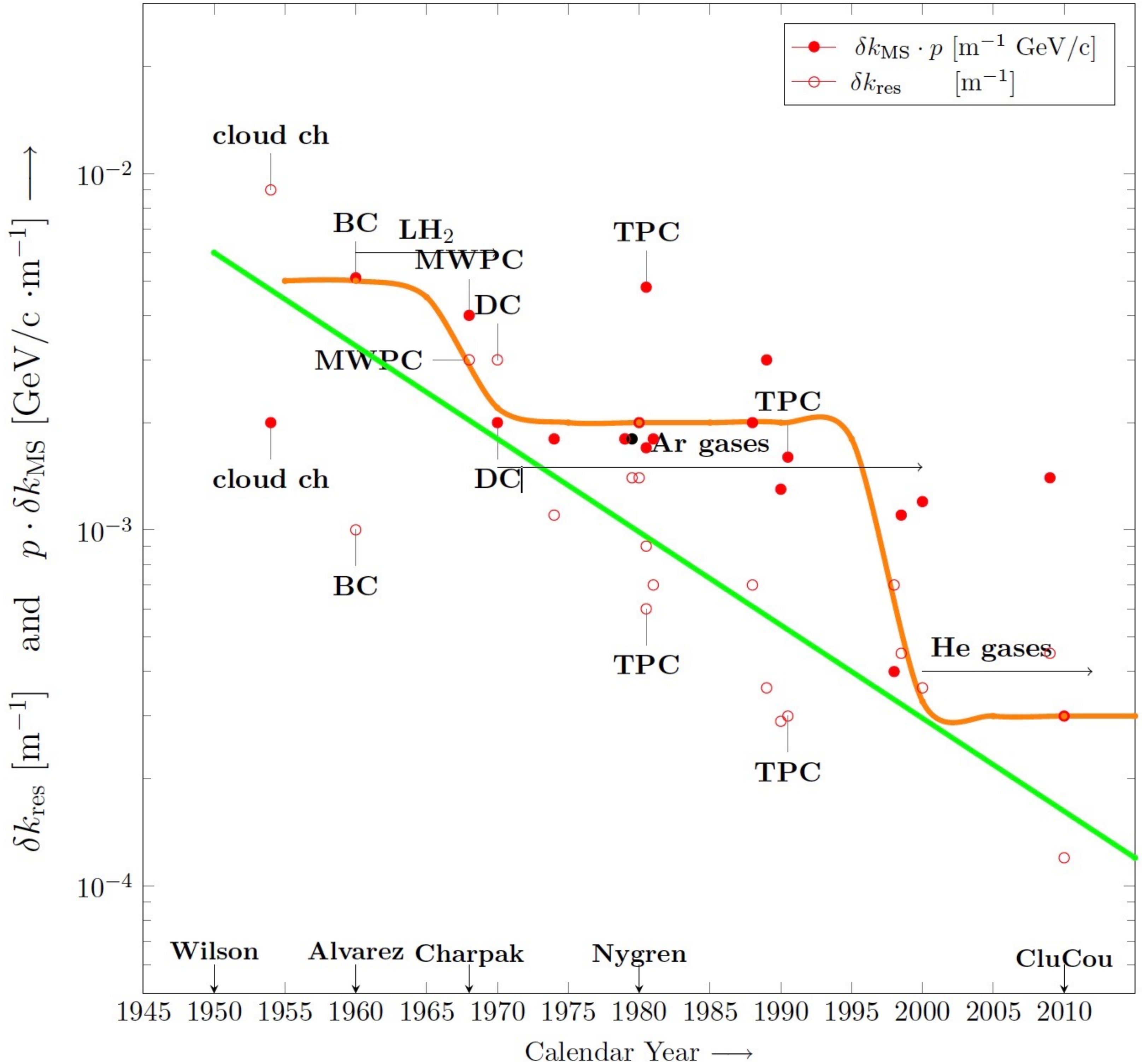


Figure 1: The curvature uncertainties due to spatial resolution fluctuations (○) $\delta k_{res} \sim \sigma_x/\ell^2 \cdot \sqrt{720/(N+4)}$, and due to multiple scattering (●) $\delta k_{MS} \sim \theta_{rms}/\ell$ at $p = 1$ GeV/c in the materials of a given tracking chamber of length $\ell = 1$m as a function of calendar year for existing chambers. The total curvature uncertainty of a track is $\delta k = \sqrt{(\delta k_{res})^2 + (\delta k_{MS})^2}$. One line shows the direct improvements in resolution, the other shows the step-wise reductions in chamber material. CluCou relies on cluster timing in a light gas mixture (*He*) proposed by Grancagnolo.